Journal of Applied Mechanics and Technical Physics, Vol. 39, No. 4, 1998

CHARGED JET OF AN INCOMPRESSIBLE LIQUID IN AN ELECTRIC FIELD

A. A. Shutov and A. A. Zakhar'yan[1]

UDC 541.24:532.5

The problem of the jet flow of an incompressible liquid with free boundaries in an electric field is solved in the approximation of a laminar boundary layer. An exact solution for a round jet is found in the class of self-similar solutions. In the case of a flat slit jet, a solution is constructed in the form of a series in powers of the coordinate transverse to the plane of symmetry. The dependence of the radius (half-width) on the longitudinal coordinate is given.

The formation of jets in an electric field is one of the sources of the generation of charged drops and filaments for various fields in electro-drop-jet technology and the production of highly efficient filtering materials [1]. The capacity to produce long jets in an electric field belongs mainly to liquids that are "poor conductors," occupying an intermediate position in electrophysical properties between insulators and electrolytes. When injected through capillaries under the action of an electric field, liquids can form surprisingly stable, very thin jets that continuous over the entire interelectrode gap, which reaches lengths of meters in individual experiments. Capillaries with diameter of ~1 mm and an electric-field strength $E \sim 10^5$ V/m or higher are used, as a rule. The resulting jets have diameter of 100–0.1 μm, and their specific volume density of charge can reach hundreds of C/m^3 [2, 3]. In the present paper, we use equations of the boundary-layer type to describe such flows, and the procedure for deriving them here does not involve the assumption of a large Reynolds number. Below, we confine our analysis of motions to the approximation of "frozen-in" charge, i.e., to large electric Reynolds numbers $\mathrm{Re}_q = V/(bE) \gg 1$, where V is the characteristic velocity, b is the charge mobility, and E is the electric-field strength [4, 5].

Space-Charged Round Jet. Let us consider a stationary axisymmetric jet with space charge density $\gamma = \text{const}$ in a uniform electric field E parallel to the z axis. The equation of motion

$$\rho \mathbf{V}\nabla\mathbf{V} = -\nabla p + \mu\Delta\mathbf{V} + \gamma\mathbf{E} \tag{1}$$

in dimensionless form contains the parameter $s = \rho Q^2/(2\pi^2\gamma E r_0^5) = \rho Q^3/(2\pi^2 I E r_0^5)$. Here V is the liquid-jet velocity, ρ and μ are the density and viscosity, Q is the volumetric flow rate of the liquid, r_0 is the initial radius of the jet, and $I = \gamma Q$ is the total electric current carried by the jet. In the case of strong fields ($s \ll 1$), which we shall consider below, the equations can be simplified as follows. We assume that the form of the continuity equation, $\partial(rv)/r\partial r + \partial u/\partial z = 0$, and the form of the kinematic condition for the stream function at the free surface, $\psi(z, r = f) = Q/2\pi$, which serves to determine the unknown boundary $r = f(z)$ do not depend on the following transformation of parameters: $z \to z$, $r \to g_1(s)r$, $u \to g_2(s)u$, and $v \to g_3(s)v$, where u and v are the longitudinal and transverse velocity, respectively. We then have the following relationship between the functions: $g_3 = g_1 g_2$ and $g_2 g_1^2 = 1$. We assume that, just as in the theory of a laminar boundary layer, the motion of liquid in a layer or a jet, going predominantly in the longitudinal direction ($v \ll u$), is due to viscous transfer of longitudinal momentum in the transverse direction. Then, from the condition of equality of the orders of magnitude with respect to s of the inertial, viscous, and electric terms in the z projection of Eq. (1), we have $g_1 \sim s^{1/4}$, $g_2 \sim s^{-1/2}$, and $g_3 \sim s^{-1/4}$. The contribution of the pressure p, which is proportional to the capillary pressure $p_T = T/f$ and is on the order of $s^{-1/4}$, is small compared to

Branch of the Karpov Physicochemistry Institute, State Science Center, Obninsk 249020. Karpov Physicochemistry Institute, State Science Center, Moscow 115523. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 39, No. 4, pp. 12–16, July–August, 1998. Original article submitted April 4, 1996; revision submitted November 28, 1996.

0021-8944/98/3904-0489 $20.00

the leading terms, which are on the order of s^{-1}. Thus, the transformation $z \to z$, $r \to s^{1/4}r$, $u \to s^{-1/2}u$, and $v \to s^{-1/4}v$ and isolation of the leading terms in powers of s^{-1} in the Navier–Stokes equations lead to the equations

$$u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r} = \frac{1}{\mathrm{Re}}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r} + \frac{1}{2s}; \tag{2}$$

$$\frac{\partial}{\partial z}ru + \frac{\partial}{\partial r}rv = 0 \tag{3}$$

with the boundary conditions

$$v(z, r = 0) = 0, \tag{4}$$

$$\psi(z, r = f) = 1/2, \tag{5}$$

$$f(z = 0) = 1. \tag{6}$$

These equations are written in dimensionless form, in which the scales r_0, $Q/(\pi r_0^2)$, and Q/π [$\mathrm{Re} = \rho Q/(\pi r_0 \mu)$] are chosen for lengths, velocities, and the stream function. Such a procedure for deriving the standard boundary-layer equations was used in [4]. Determining ψ from the equations $u = \partial\psi/r\partial r$ and $v = -\partial\psi/r\partial z$ and taking $\psi(z, r) = \Phi(\xi)z$, where $\xi = r^2/\sqrt{z}$, from (2) we obtain the following equation for Φ:

$$8(\xi\Phi'')'/\mathrm{Re} + 4\Phi\Phi'' - 2\Phi'^2 + 1/2s = 0 \tag{7}$$

(a prime denotes differentiation with respect to ξ). The solution of Eq. (7) in the form of a series in whole powers of the argument, which because of condition (4) begins with the linear term, is cut off at the third term, and for nontrivial coefficients of the function $\Phi = a_1\xi + a_2\xi^2$, we have the equation $16a_2/\mathrm{Re} - 2a_1^2 + 1/2s = 0$. Substituting this solution into (5), we find the dependence of the radius of the jet on the longitudinal coordinate:

$$f(z) = \sqrt{\frac{a_1\sqrt{z}}{2a_2}}\sqrt{\sqrt{1 + \frac{2a_2}{a_1^2 z}} - 1}. \tag{8}$$

From condition (6) we find $a_2 = 1/2$, so that

$$a_1 = \sqrt{\frac{1}{4s} + \frac{4}{\mathrm{Re}}}.$$

For $z \gg 1$ and $s \ll \mathrm{Re}/16$, from (8) we obtain an expression for the radius,

$$f(z) = (s/z)^{1/4}, \tag{9}$$

which in dimensional form does not depend on the initial radius or the viscosity of the liquid.

Surface-Charged Round Jet. The electric and hydrodynamic fields interact through the jet boundary: the force σE acts per unit surface area in the tangential direction and the force $\sigma^2/2\varepsilon_0$ in the normal direction, neglecting the dielectric term (σ is the surface charge density and ε_0 is the dielectric constant) [6]. In accordance with the condition of constancy of the electric current of "frozen-in" charges in the deformed volume, we have $\sigma \sim f(z)$. As the jet contracts, σ decreases, so that the normal action on its boundary, which is a quadratic function of σ, can produce a dominant pressure field in the overall balance of interactions only in the immediate vicinity of the liquid injection point. Downstream the dynamics is determined by the transfer of longitudinal momentum across the jet from the boundary toward the axis. A similar mechanism of flow development by relaxation of longitudinal momentum into the interior of the liquid occurs in a submerged jet, in which, in contrast to the case under consideration, the viscous transfer of momentum goes in the opposite direction, from the axis toward the periphery. Let us consider the equations of a laminar boundary layer (2) and (3) for $\gamma = 0$, which are commonly used in the theory of submerged jets [7, 8], with Eqs. (4) and (5) and the boundary condition for shear stress. We obtain the latter from the following considerations. For a conserved electric current I, we have $I = \sigma 2\pi f v_r = 2\sigma Q/f$, where v_r is the velocity of the boundary streamline. Substituting σ into the tangential condition $\mu(\partial u/\partial r + \partial v/\partial z) = \sigma E$ and isolating the leading terms in s, we obtain the following boundary condition in the tangential direction:

$$\frac{1}{f}\frac{\partial u}{\partial r} = \frac{\text{Re}}{4s} \qquad \text{for} \quad r = f(z). \tag{10}$$

The solution of problem (2)–(4) and (10) for $\gamma = 0$ is found in a domain limited in the transverse coordinate to the radius of the jet, $r \leqslant f(z)$, $0 \leqslant z < \infty$, where $f(z)$ must also be determined in the course of the solution. The procedure used above to determine $\psi = \Phi(\xi)z$ and $\xi = r^2/\sqrt{z}$, yields the following expressions for the surface of a charged jet:

$$\Phi(\xi) = c_1\xi + c_2\xi^2, \quad c_1 = 1/\sqrt{4s}, \quad c_2 = \text{Re}/32s, \quad f(z) = \left(\frac{64sz}{\text{Re}^2}\right)^{1/4}\sqrt{\sqrt{1+\frac{\text{Re}}{4z}} - 1}. \tag{11}$$

Dependence (11) for $z \gg \text{Re}/4$ coincides with (9). Relations (9) and (11) agree well with experimental profiles in the region far from the liquid injection point [9, 10], whereas they give an incorrect coordinate dependence of the jet radius near the coordinate origin. The latter is due to the fact that the Ohmic component in the overall charge flow plays an important role in this region, and the assumption that the charge is frozen in is therefore invalid.

Plane, Surface-Charged Jet. The formulation of the jet problem for a liquid escaping from a plane-parallel slit lying in the yz plane, analogous to the formulation for a round jet, is represented in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\text{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right); \tag{12}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \tag{13}$$

$$\lim_{x\to\infty}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\text{Re}}{s}f \qquad \text{for} \quad y = f(x); \tag{14}$$

$$v(x, y = 0) = 0; \tag{15}$$

$$\psi(x, y = f(x)) = 1, \qquad |y| \leqslant f(x), \qquad 0 \leqslant x < \infty, \tag{16}$$

where lengths, velocities, and the stream function are normalized to the quantities L, q/L, and q, respectively, u and v are the velocities in the longitudinal (x) and transverse (y) directions, and L is the unit of length in the z direction. The quantities $q = \Delta Q/L$ and $i = \Delta I/L$ are the specific flow rate and the current per unit length, respectively. Here ΔQ and ΔI are the volumetric flow rate and the electric current along a strip of length L cut out from the jet by two planes perpendicular to the z axis. We assume that the velocities do not depend on the z coordinate, the field E is directed along the x axis, $f(x)$ is the half-thickness of the jet, $\text{Re} = \rho q/\mu$, and $s = \rho q^3/(iEL^3)$. Boundary condition (14) is written in the form of a limiting transition because of the assumption that charge transfer is purely convective, which is satisfied the more accurately, the farther an element of the jet lies from the coordinate origin. Just as in the axisymmetric problem, we seek a solution that gives an expression for u that increases along the longitudinal coordinate. We construct the solution of problem (12)–(16) in the form of a power series with respect to the transverse coordinate, $u = \sum\limits_0 b_n y^n$, where b_n is a function only of x. We use the symmetry of the velocity, $u(x,y) = u(x,-y)$, and the expansion of u then contains only even powers of y. We write velocities satisfying the continuity equation (13) and condition (15) as follows:

$$u = \sum_0 b_{2n}y^{2n}, \qquad v = -\sum_0 b'_{2n}(x)y^{2n+1}/(2n+1).$$

Substitution of these expressions into (12) and isolation of terms to the same powers of y lead to a finite system of equations that has the following form for the first four functions b_n:

$$b_0b'_0 = \frac{b''_0 + 2b_2}{\text{Re}}, \qquad b_0b'_2 - b'_0b_2 = \frac{b''_2 + 12b_4}{\text{Re}}, \qquad b_0b'_4 - 3b'_0b_4 + b_2b'_2/3 = \frac{b''_4 + 30b_6}{\text{Re}}. \tag{17}$$

This system is constructed so that with the successive addition to it of new equations, only one new unknown is introduced each time in the current list of functions b_n to be determined. This considerably simplifies the procedure for solving system (17). Indeed, substituting the velocity into (14), we have

$$\lim_{x\to\infty}[2b_2 - b_0'' + (4b_4 - b_2''/3)f^2 + (6b_6 - b_4''/5)f^4 + \ldots] = \mathrm{Re}/s. \tag{18}$$

As shown below, $f(x)$ and all the coefficients b_n except for b_0, are decreasing functions of x, and then from (18) we obtain

$$2b_2 - b_0'' = \mathrm{Re}/s. \tag{19}$$

Eliminating b_2 from (19) and the first equation of system (17), we obtain the following equation for b_0:

$$b_0 b_0' = \frac{2b_0'' + \mathrm{Re}/s}{\mathrm{Re}}.$$

Integrating it, we have $b_0' - \alpha b_0^2 + \beta x = 0$, where $\alpha = \mathrm{Re}/4$, and $\beta = \mathrm{Re}/2s$. The change of variables $b_0 = -g'/\alpha g$ and $t = (\alpha\beta)^{1/3}x$ converts this equation into $g'' - gt = 0$, whose solution is the Airy function $g = c_1 Ai(t) + c_2 Bi(t)$ [11]. Since $Bi(t)$ results in a slowing flow, we have the constant $c_2 = 0$. Returning to the variables b_0 and x, we obtain

$$b_0 = \sqrt{\frac{\beta x}{\alpha}}\frac{K_{2/3}(\eta)}{K_{1/3}(\eta)},$$

where K_m is a MacDonald function of the mth order; $\eta = 2(\alpha\beta)^{1/2}x^{3/2}/3$. All the remaining coefficients b_n are determined by simple differentiation of the corresponding expressions; in particular, we find b_2 using (19), b_4 is calculated from the second equation of system (17), etc. An idea of the behavior of b_n at large x is given by the first terms of the expansions of the corresponding functions:

$$b_0 = \sqrt{\beta x/\alpha} - 1/2\alpha x, \quad b_2 = \mathrm{Re}/2s - \sqrt{\beta/64\alpha x^3}, \quad b_4 = \frac{\mathrm{Re}^2}{48s}\sqrt{\frac{\beta}{\alpha x}}.$$

We note that the dependence $u = \sqrt{x}(1 + \mathrm{const}\cdot y^2/\sqrt{x})$ is similar to the corresponding dependence for a round jet, whereas the asymptotic profile in the flat case, $f(x) = \sqrt{s/2x}$, decreases faster along the longitudinal coordinate than that for a round jet.

REFERENCES

1. I. V. Petryanov, V. I. Kozlov, P. I. Basmanov, and B. I. Ogorodnikov, *Filamentary Filtering Materials* [in Russian], Znanie, Moscow (1968).
2. S. I. Popov and I. V. Petryanov, "On the mechanism of electrostatic spraying of liquids," *Dokl. Akad. Nauk SSSR*, **195**, No. 4, 893–895 (1970).
3. V. N. Kirichenko, A. D. Shepelev, and V. N. Polevov, "Specific charge of a liquid in electrohydrodynamic processes of spraying and the formation of microfilaments," *Dokl. Akad. Nauk SSSR*, **315**, No. 4, 819–823 (1990).
4. L. I. Sedov, *Mechanics of a Continuous Medium* [in Russian], Vols. 1 and 2, Nauka, Moscow (1983).
5. V. V. Gogosov, V. A. Polyanskii, I. P. Semenova, and A. E. Yakubenko, "Equations of electrohydrodynamics and transfer coefficients in a strong electric field," *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, No. 2, 31–45 (1969).
6. I. E. Tamm, *Principles of the Theory of Electricity* [in Russian], Gostekhteoretizdat, Moscow (1957).
7. O. G. Martynenko, V. N. Korovkin, and Yu. A. Sokovishin, *Theory of Laminar Viscous Jets* [in Russian], Nauka i Tekhnika, Minsk (1985).
8. L. G. Loitsyanskii, *Mechanics of Liquids and Gases*, Pergamon Press, Oxford-New York (1966).
9. P. K. Baumgarten, "Electrostatic spinning of acrylic microfibers," *J. Colloid Interface Sci.*, **36**, No. 1, 71–79 (1971).
10. A. A. Shutov, "Shape of an incompressible, weakly conducting jet in a strong electric field," *Prikl. Mekh. Tekh. Fiz.*, No. 2, 20–25 (1991).
11. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, John Wiley and Sons, New York (1972).